Integration durch Partialbruchzerlegung

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__Beispiel:__ $\frac{2}{x-1} + \frac{3}{x+5} = \frac{2(x+5)+3(x-1)}{(x-1)\cdot(x+5)} = \frac{5x+7}{(x-1)(x+5)}$

Umformen des Ansatzes:

$$\frac{5x+7}{(x-1)(x+5)} = \frac{a}{x-1} + \frac{b}{x+5} \qquad \Big|\cdot(x-1)(x+5)$$

$$5x+7 = a(x+5) + b(x-1)$$

$$5x+7 = (a+b)\cdot x + (5a-b)$$

LGS:

$$\begin{aligned} a+b &= 5 && (1)\\ 5a-b &= 7 && (2)\\ \hline (1)+(2) \quad 6a &= 12 \\ a &= 2 && \text{in } (1)\\ b &= 3 \end{aligned}$$

Partialbruchzerlegung: $\frac{5x+7}{(x-1)(x+5)} = \frac{2}{x-1} + \frac{3}{x+5}$

$$\int_2^8 \frac{5x+7}{(x-1)(x+5)}\,dx = \int_2^8 \frac{2}{x-1} + \frac{3}{x+5}\,dx = \Big[2\ln|x-1| + 3\ln|x+5|\Big]_2^8$$

$$= 2\ln 7 + 3\ln 13 - 2\ln 1 - 3\ln 7$$

$$= -\ln 7 + 3\ln 13$$

__Anwendung__

2. $\frac{x+1}{(x-1)^2} = \frac{a}{x-1} + \frac{b}{x-1} \qquad \Big|\cdot(x-1)^2$

$$x+1 = a(x-1) + b(x-1)$$

$$x+1 = (a+b)x + (-a-b)$$

LGS:

$$\begin{aligned} a+b &= 1\\ -a-b &= 1\\ \hline 0 &= 1 \end{aligned}$$

Die Parameter a und b werden mit demselben Linearfaktor (x-1) multipliziert. Somit sind die linken Seiten des entstehenden LGS Vielfache voneinander. Damit hat dieses LGS keine Lösung. (Der Fall „unendlich viele Lösungen" würde nur auftreten, wenn im Zähler des ursprünglichen Bruchs ebenfalls $(x-1)$ stehen würde.)

Verändern des Ansatzes:

$$f(x) = \frac{x+1}{(x-1)^2} = \frac{x-1+2}{(x-1)^2} = \frac{x-1}{(x-1)^2} + \frac{2}{(x-1)^2} = \frac{1}{x-1} + \frac{2}{(x-1)^2}$$

$$F(x) = \ln|x-1| - \frac{2}{x-1}$$

4. Polynomdivision

$$(2x^3 - 4x^2 + 14x - 2) : (x^2 - 4x + 12) = 2x + 4 + \frac{6x-50}{x^2-4x+12}$$

$$\begin{array}{l} -(2x^3 - 8x^2 + 24x) \\ \hline \qquad 4x^2 - 10x - 2 \\ \quad -(4x^2 - 16x + 48) \\ \hline \qquad\qquad 6x - 50 \end{array}$$

Aufgaben

1a) $\frac{3x+3}{(x-2)(x+7)} = \frac{a}{x-2} + \frac{b}{x+7} \quad |\cdot(x-2)(x+7)$

$3x+3 = (a+b)x + (7a-2b)$

LGS: $\begin{Bmatrix} a+b=3 \\ 7a-2b=3 \end{Bmatrix} \Leftrightarrow \begin{Bmatrix} 9a=9 \\ a+b=3 \end{Bmatrix} \Leftrightarrow \begin{Bmatrix} a=1 \\ b=2 \end{Bmatrix}$

$$\int_{-1}^{1} \frac{3x+3}{(x-2)(x+7)}\,dx = \int_{-1}^{1} \frac{1}{x-2} + \frac{2}{x+7}\,dx = \left[\ln|x-2| + 2\ln|x+7|\right]_{-1}^{1}$$

$$= \ln 1 + 2\ln 8 - \ln 3 - 2\ln 6$$

$$= 4\ln 2 - 3\ln 3$$

b) $\frac{2}{(x-4)(x+1)} = \frac{a}{x-4} + \frac{b}{x+1} \quad |\cdot(x-4)(x+1)$

$2 = (a+b)x + (a-4b)$

LGS: $\begin{Bmatrix} a+b=0 \\ a-4b=2 \end{Bmatrix} \Leftrightarrow \begin{Bmatrix} 5b=-2 \\ a=-b \end{Bmatrix} \Leftrightarrow \begin{Bmatrix} b=-0{,}4 \\ a=0{,}4 \end{Bmatrix}$

$$\int_5^6 \frac{2}{(x-4)(x+1)}\,dx = \int_5^6 \frac{0{,}4}{x-4} - \frac{0{,}4}{x+1}\,dx = \Big[0{,}4\ln|x-4| - 0{,}4\ln|x+1|\Big]_5^6$$

$$= 0{,}4\ln 2 - 0{,}4\ln 7 - 0{,}4\ln 1 + 0{,}4\ln 6$$

$$= 0{,}4(2\ln 2 - \ln 7 + \ln 3)$$

c) $$\frac{2x+2}{x(x-1)(x-2)} = \frac{a}{x} + \frac{b}{x-1} + \frac{c}{x-2} \qquad |\cdot x(x-1)(x-2)$$

$$2x+2 = a(x^2-3x+2) + b(x^2-2x) + c(x^2-x)$$

$$2x+2 = (a+b+c)x^2 + (-3a-2b-c)x + 2a$$

LGS: $$\left\{\begin{aligned} a+b+c &= 0 \\ -3a-2b-c &= 2 \\ 2a &= 2 \end{aligned}\right\} \Leftrightarrow \left\{\begin{aligned} a &= 1 \\ b+c &= -1 \\ -2b-c &= 5 \end{aligned}\right\} \Leftrightarrow \left\{\begin{aligned} a &= 1 \\ b &= -4 \\ c &= 3 \end{aligned}\right\}$$

$$\int_3^4 \frac{2x+2}{x(x-1)(x-2)}\,dx = \int_3^4 \frac{1}{x} - \frac{4}{x-1} + \frac{3}{x-2}\,dx = \Big[\ln|x| - 4\ln|x-1| + 3\ln|x-2|\Big]_3^4$$

$$= \ln 4 - 4\ln 3 + 3\ln 2 - \ln 3 + 4\ln 2 - 3\ln 1$$

$$= 9\ln 2 - 5\ln 3$$

2a) $$\frac{2x-4}{(x-3)^2} = \frac{a}{x-3} + \frac{b}{(x-3)^2} \qquad |\cdot(x-3)^2$$

$$2x-4 = a(x-3) + b = ax + (-3a+b)$$

LGS: $$\left\{\begin{aligned} a &= 2 \\ -3a+b &= -4 \end{aligned}\right\} \Leftrightarrow \left\{\begin{aligned} a &= 2 \\ b &= 2 \end{aligned}\right\}$$

$$\int_0^1 \frac{2x-4}{(x-3)^2}\,dx = \int_0^1 \frac{2}{x-3} + \frac{2}{(x-3)^2}\,dx = \Big[2\ln|x-3| - \frac{2}{x-3}\Big]_0^1$$

$$= 2\ln 2 + 1 - 2\ln 3 + \frac{2}{3} = 2\ln 2 - 2\ln 3 + \frac{5}{3}$$

b) $$\frac{2x^2+3x-2}{x^2(x-1)} = \frac{a}{x} + \frac{b}{x^2} + \frac{c}{x-1} \qquad |\cdot x^2(x-1)$$

$$2x^2+3x-2 = a\cdot(x^2-x) + b(x-1) + cx^2$$

$$2x^2+3x-2 = (a+c)x^2 + (-a+b)x - b$$

$$\text{LGS: } \left\{\begin{array}{l} a + c = 2 \\ -a + b = 3 \\ -b = -2 \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} b = 2 \\ a = -1 \\ c = 3 \end{array}\right\}$$

$$\int_2^4 \frac{2x^2+3x-2}{x^2(x-1)}\,dx = \int_2^4 -\frac{1}{x} + \frac{2}{x^2} + \frac{3}{x-1}\,dx = \Big[-\ln|x| - \frac{2}{x} + 3\ln|x-1|\Big]_2^4$$

$$= -\ln 4 - \frac{1}{2} + 3\ln 3 + \ln 2 + 1 - 3\ln 1$$

$$= -\ln 2 + 3\ln 3 + \frac{1}{2}$$

c) $$\frac{x^2-1}{x(x-2)^2} = \frac{a}{x} + \frac{b}{x-2} + \frac{c}{(x-2)^2} \qquad \Big| \cdot x(x-2)^2$$

$$x^2 - 1 = a(x^2-4x+4) + b(x^2-2x) + cx$$

$$x^2 - 1 = (a+b)x^2 + (-4a-2b+c)x + 4a$$

$$\text{LGS: } \left\{\begin{array}{l} a + b = 1 \\ -4a - 2b + c = 0 \\ 4a = -1 \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} a = -\frac{1}{4} \\ b = \frac{5}{4} \\ c = \frac{3}{2} \end{array}\right\}$$

$$\int_{-2}^{-1} \frac{x^2-1}{x(x-2)^2}\,dx = \int_{-2}^{-1} -\frac{1}{4x} + \frac{5}{4(x-2)} + \frac{3}{2(x-2)^2}\,dx$$

$$= \Big[-\frac{1}{4}\ln|x| + \frac{5}{4}\ln|x-2| - \frac{3}{2(x-2)}\Big]_{-2}^{-1}$$

$$= -\frac{1}{4}\ln 1 + \frac{5}{4}\ln 3 + \frac{1}{2} + \frac{1}{4}\ln 2 - \frac{5}{4}\ln 4 - \frac{3}{8}$$

$$= \frac{5}{4}\ln 3 - \frac{9}{4}\ln 2 + \frac{1}{8}$$

3a) $x^2 - 3x - 10 = 0$

$(x+2)(x-5) = 0$

$$\frac{7x+7}{(x+2)(x-5)} = \frac{a}{x+2} + \frac{b}{x-5}$$

$$7x + 7 = (a+b)x + (-5a+2b)$$

$$\text{LGS: } \left\{\begin{array}{l} a + b = 7 \\ -5a + 2b = 7 \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} 7b = 42 \\ a + b = 7 \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} b = 6 \\ a = 1 \end{array}\right\}$$

$$\ldots + 6\ln|x-5|\Big]^{2}$$

$$\int_1^2 \frac{7x+7}{x^2-3x-10}\,dx = \int_1^2 \frac{1}{x+2} + \frac{6}{x-5}\,dx = \left[\ln|x+2| + 6\ln|x-5|\right]_1^2$$

$$= \ln 4 + 6\ln 3 - \ln 3 - 6\ln 4$$

$$= -5\ln 4 + 5\ln 3$$

b) $4 - x^2 = 0$

$(2-x)(2+x) = 0$

$$\frac{6}{4-x^2} = \frac{a}{2-x} + \frac{b}{2+x} \qquad |\cdot(4-x^2)$$

$$6 = a(2+x) + b(2-x)$$

$$6 = (a-b)x + (2a+2b)$$

LGS: $\begin{Bmatrix} a-b=0 \\ 2a+2b=6 \end{Bmatrix} \iff \begin{Bmatrix} a=b \\ 4a=6 \end{Bmatrix} \iff \begin{Bmatrix} a=1{,}5 \\ b=1{,}5 \end{Bmatrix}$

$$\int_0^1 \frac{6}{4-x^2}\,dx = \int_0^1 \frac{1{,}5}{2-x} + \frac{1{,}5}{2+x}\,dx = \left[-1{,}5\ln|2-x| + 1{,}5\ln|2+x|\right]_0^1$$

$$= -1{,}5\ln 1 + 1{,}5\ln 3 + 1{,}5\ln 2 - 1{,}5\ln 2$$

$$= 1{,}5\ln 3$$

c) $x^3 - 6x^2 + 9x = 0$

$x(x-3)^2 = 0$

$$\frac{27-x-2x^2}{x^3-6x^2+9x} = \frac{a}{x} + \frac{b}{x-3} + \frac{c}{(x-3)^2} \qquad |\cdot x(x-3)^2$$

$$-2x^2 - x + 27 = a(x^2-6x+9) + bx(x-3) + cx$$

$$-2x^2 - x + 27 = (a+b)x^2 + (-6a-3b+c)x + 9a$$

LGS: $\begin{Bmatrix} 9a = 27 \\ -6a-3b+c = -1 \\ a+b = -2 \end{Bmatrix} \iff \begin{Bmatrix} a=3 \\ b=-5 \\ c=2 \end{Bmatrix}$

$$\int_1^2 \frac{27-x-2x^2}{x^3-6x^2+9x}\,dx = \int_1^2 \frac{3}{x} - \frac{5}{x-3} + \frac{2}{(x-3)^2}\,dx$$

$$= \left[3\ln|x| - 5\ln|x-3| - \frac{2}{x-3}\right]_1^2$$

$$= 3\ln 2 - 5\ln|-1| - \frac{2}{-1} - 3\ln 1 + 5\ln|-2| + \frac{2}{-2}$$

$$= 8\ln 2 + 1$$

d) $x^3 - 8x^2 + 12x = 0$

$x(x-2)(x-6) = 0$

$$\frac{-24x+24}{x^3-8x^2+12x} = \frac{a}{x} + \frac{b}{x-2} + \frac{c}{x-6} \qquad |\cdot x(x-2)(x-6)$$

$$-24x+24 = a(x^2-8x+12) + b(x^2-6x) + c(x^2-2x)$$

$$-24x+24 = (a+b+c)x^2 + (-8a-6b-2c)x + 12a$$

$$\text{LGS:} \left\{\begin{array}{l} 12a = 24 \\ -8a-6b-2c = -24 \\ a+b+c = 0 \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} a = 2 \\ -6b-2c = -8 \\ b+c = -2 \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} a = 2 \\ b = 3 \\ c = -5 \end{array}\right\}$$

$$\int_3^5 \frac{-24x+24}{x^3-8x^2+12x}\,dx = \int_3^5 \frac{2}{x} + \frac{3}{x-2} - \frac{5}{x-6}\,dx = \left[2\ln x + 3\ln(x-2) - 5\ln|x-6|\right]_3^5$$

$$= 2\ln 5 + 3\ln 3 - 5\ln|1| - 2\ln 3 - 3\ln 1 + 5\ln|-3|$$

$$= 2\ln 5 + 6\ln 3$$

e) $x^3 - 3x - 2 = 0$

$(x^3 - 3x - 2) : (x+1) = x^2 - x - 2 = (x-2)(x+1)$

$-(x^3 + x^2)$

$\quad -x^2 - 3x$

$\quad -(-x^2 - x)$

$\qquad -2x - 2$

$\qquad -(-2x-2)$

$\qquad\quad 0$

$\Rightarrow x^3 - 3x - 2 = (x+1)^2(x-2)$

$$\frac{9x^2+9x+9}{x^3-3x-2} = \frac{a}{x+1} + \frac{b}{(x+1)^2} + \frac{c}{x-2} \qquad |\cdot (x+1)^2(x-2)$$

$$9x^2+9x+9 = a(x^2-x-2) + b(x-2) + c(x^2+2x+1)$$

$$9x^2+9x+9 = (a+c)x^2 + (-a+b+2c)x + (-2a-2b+c)$$

LGS: $a + c = 9 \qquad a + c = 9$

$$\text{LGS:} \left\{\begin{array}{l} a \qquad\quad + c = 9 \\ -a + b + 2c = 9 \\ -2a - 2b + c = 9 \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} a \qquad\quad + c = 9 \\ -4a \qquad + 5c = 27 \\ -a + b + 2c = 9 \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} c = 7 \\ a = 2 \\ b = -3 \end{array}\right\}$$

$$\int_0^1 \frac{9x^2 + 9x + 9}{x^3 - 3x - 2}\,dx = \int_0^1 \frac{2}{x+1} - \frac{3}{(x+1)^2} + \frac{7}{x-2}\,dx$$

$$= \left[2\ln|x+1| + \frac{3}{x+1} + 7\ln|x-2|\right]_0^1$$

$$= 2\ln 2 + \frac{3}{2} + 7\ln|-1| - 2\ln 1 - \frac{3}{1} - 7\ln|-2|$$

$$= -5\ln 2 - \frac{3}{2}$$

f) $x^3 - 5x^2 + 7x - 3 = 0$

$$(x^3 - 5x^2 + 7x - 3) : (x-1) = x^2 - 4x + 3 = (x-1)(x-3)$$

$$\begin{array}{l} -(x^3 - x^2) \\ \hline \quad -4x^2 + 7x \\ \quad -(-4x^2 + 4x) \\ \hline \qquad 3x - 3 \\ \qquad -(3x - 3) \\ \hline \qquad\quad 0 \end{array}$$

$$\Rightarrow x^3 - 5x^2 + 7x - 3 = (x-1)^2(x-3)$$

$$\frac{2(x^2+1)}{x^3 - 5x^2 + 7x - 3} = \frac{a}{x-1} + \frac{b}{(x-1)^2} + \frac{c}{x-3} \qquad | \cdot (x-1)^2(x-3)$$

$$2x^2 + 2 = a(x^2 - 4x + 3) + b(x-3) + c(x^2 - 2x + 1)$$

$$2x^2 + 2 = (a+c)x^2 + (-4a + b - 2c)x + (3a - 3b + c)$$

$$\text{LGS:} \left\{\begin{array}{l} a \qquad\quad + c = 2 \\ -4a + b - 2c = 0 \\ 3a - 3b + c = 2 \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} a \qquad\quad + c = 2 \\ -9a \qquad - 5c = 2 \\ 3a - 3b + c = 2 \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} a = -3 \\ c = 5 \\ b = -2 \end{array}\right\}$$

$$\int_4^6 \frac{2(x^2+1)}{x^3 - 5x^2 + 7x - 3}\,dx = \int_4^6 -\frac{3}{x-1} - \frac{2}{(x-1)^2} + \frac{5}{x-3}\,dx$$

$$= \left[-3\ln|x-1| + \frac{2}{x-1} + 5\ln|x-3|\right]_4^6$$

$$= -3\ln 5 + \frac{2}{5} + 5\ln 3 + 3\ln 3 - \frac{2}{3} - 5\ln 1$$

$$= -3\ln 5 + 8\ln 3 - \frac{4}{15}$$

4) a) $(x^2+4):(x^2-4) = 1 + \frac{8}{x^2-4} = 1 + \frac{8}{(x-2)(x+2)}$

$$\begin{array}{l} -(x^2-4) \\ \hline \quad 8 \end{array}$$

$$\frac{8}{x^2-4} = \frac{a}{x-2} + \frac{b}{x+2} \qquad | \cdot (x-2)(x+2)$$

$$8 = (a+b)x + (2a-2b)$$

LGS: $\left\{ \begin{array}{l} a+b=0 \\ 2a-2b=8 \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} a=2 \\ b=-2 \end{array} \right\}$

$$\int_3^5 \frac{x^2+4}{x^2-4}\,dx = \int_3^5 1 + \frac{2}{x-2} - \frac{2}{x+2}\,dx$$

$$= \Big[x + 2\ln|x-2| - 2\ln|x+2|\Big]_3^5$$

$$= 5 + 2\ln 3 - 2\ln 7 - 3 - 2\ln 1 + 2\ln 5$$

$$= 2 + 2\ln 3 - 2\ln 7 + 2\ln 5$$

b) $(4x^2-2x+2):(x^2-4x+3) = 4 + \frac{14x-10}{x^2-4x+3} = 4 + \frac{14x-10}{(x-1)(x-3)}$

$$\begin{array}{l} -(4x^2-16x+12) \\ \hline \quad 14x-10 \end{array}$$

$$\frac{14x-10}{x^2-4x+3} = \frac{a}{x-1} + \frac{b}{x-3} \qquad | \cdot (x-1)(x-3)$$

$$14x-10 = (a+b)x + (-3a-b)$$

LGS: $\left\{ \begin{array}{l} a+b=14 \\ -3a-b=-10 \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} a=-2 \\ b=16 \end{array} \right\}$

$$\int_{-1}^0 \frac{4x^2-2x+2}{x^2-4x+3}\,dx = \int_{-1}^0 4 - \frac{2}{x-1} + \frac{16}{x-3}\,dx = \Big[4x - 2\ln|x-1| + 16\ln|x-3|\Big]_{-1}^0$$

$$= 0 - 2\ln|-1| + 16\ln|-3| + 4 + 2\ln|-2| - 16\ln|-4|$$

$$= 16\ln 3 - 30\ln 2 + 4$$

c) $(x^3-8x^2+14x-2):(x^2-8x+12) = x + \frac{2x-2}{x^2-8x+12} = x + \frac{2x-2}{(x-2)(x-6)}$

$$\underline{-(x^3-8x^2+12x)}$$
$$2x-2$$

$$\frac{2x-2}{x^2-8x+12} = \frac{a}{x-2} + \frac{b}{x-6} \qquad |(x-2)(x-6)$$

$$2x-2 = (a+b)x + (-6a-2b)$$

LGS: $\left\{\begin{matrix} a+b = 2 \\ -6a-2b = -2 \end{matrix}\right\} \Leftrightarrow \left\{\begin{matrix} a = -\frac{1}{2} \\ b = \frac{5}{2} \end{matrix}\right\}$

$$\int_0^1 \frac{x^3-8x^2+14x-2}{x^2-8x+12}dx = \int_0^1 x - \frac{1}{2}\cdot\frac{1}{x-2} + \frac{5}{2}\cdot\frac{1}{x-6}\,dx$$

$$= \left[\frac{1}{2}x^2 - \frac{1}{2}\ln|x-2| + \frac{5}{2}\ln|x-6|\right]_0^1$$

$$= \frac{1}{2} - \frac{1}{2}\ln|-1| + \frac{5}{2}\ln|-5| - 0 + \frac{1}{2}\ln|-2| - \frac{5}{2}\ln|-6|$$

$$= \frac{1}{2} + \frac{5}{2}\ln 5 - 2\ln 2 - \frac{5}{2}\ln 3$$